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ESTIMATION OF ADDED-MASS AND DAMPING COEFFICIENTS OF A TETHERED SPHERICAL FLOAT USING POTENTIAL FLOW THEORY

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Abstract—Added-mass (α) and damping coefficients (β) of a tethered spherical float, undergoing oscillatory motion in sinusoidal waves, have been derived from the motion generated velocity potential for one degree-of-freedom (surge) using potential flow theory. Variations of added-mass and damping coefficients with respect to water depth, wave period and float size have been obtained. Variations of added-mass and damping coefficients with wave period show that these hydrodynamic coefficients are frequency dependent, and this is because motion generated velocity potential is associated with the generation of surface gravity waves by the oscillating float. The added-mass and damping coefficients are unaffected by water depth, but increase with size of the float. Natural and resonant frequencies of the tethered float are computed for a wide range of wave periods using α and β values. Theoretical values of natural period of oscillation of the float agree closely with the experimental values.

NOMENCLATURE

A_o	wave amplitude
c_o	immersed portion of the cross-sectional contour of the float
d	water depth
d_s	depth of submergence of float
ds	elemental area
F_M	hydrodynamic force due to surge motion
f_w	amplitude of wave force
g	acceleration due to gravity
H	wave height
j	$\sqrt{-1}$
k	wave number
l	tether length
m	mass of the float
n_1	x-component of unit normal vector
R	radius of the float
T	wave period
T_n, T_r	natural and resonant periods of the float, respectively
t	time
x, $\dot{x}$, $\ddot{x}$	float displacement, velocity and acceleration, respectively
x_o	complex amplitude of float displacement
α	added-mass
β	damping coefficient

λ	wave length
ρ	sea water mass density
ϕ_M	motion generated velocity potential
ω	radian frequency
ω_n, ω_r	natural and resonant radian frequencies of the float, respectively.

1. INTRODUCTION

THE NEED for accurate estimation of hydrodynamic forces for the economic design of offshore structures has created considerable mathematical and experimental interest in ocean engineering. Added-mass (α) and damping coefficients (β) are important parameters to determine the amplitude of motion of any offshore structure. Dynamic characteristics of the offshore structures could be well studied if these hydrodynamic coefficients are accurately predicted. The added-mass represents the entrained mass of water surrounding the body which moves with the submerged body. It depends on the size and shape of the body, direction of motion, wave period and boundary conditions.

Considerable work has been carried out in previous years to compute added-mass and damping coefficients of particular body forms as functions of frequency. For example, Kim (1965) has determined α and β for a semi-ellipsoidal body with its origin on the free surface in water of infinite depth. Chung (1976) has developed a method to compute hydrodynamic forces for floating bodies using potential flow theory. Frank (1967) conducted a series of experiments to determine the two-dimensional added-mass and damping coefficients including the free surface effect and to check the validity of the prediction by potential flow theory. Newton *et al.* (1974) and Newton (1975) have presented FEM solution for these coefficients for hull forms of body. Chung (1977) has presented added-mass and damping coefficients as functions of frequency, direction of oscillation and depth of submergence. In his study the body was forced to sway and heave sinusoidally with small amplitudes for several depths of submergence. Hanif (1983) determined these hydrodynamic coefficients using FEM and compared his results with other investigators. In the present study, added-mass and damping coefficients of a tethered spherical float are determined using potential flow theory.

2. FORMULATION OF THE PROBLEM

A single tethered spherical float, undergoing oscillatory motion in an inviscid and incompressible fluid and irrotational flow has been considered for the theoretical analysis. Linearity is assumed throughout the analysis. The analysis has been carried out in two-dimensional cartesian coordinate system with the xy-plane parallel to the sea surface. Due to float–fluid interaction, the following three forces can be expected: (1) the pressure force exerting on the float by the incident waves, represented by incident wave potential, (2) the force resulting from the disturbance of the incident waves by the body, represented by diffraction potential and (3) the force resulting from the motion of the body computed as though it undergoes the same motion in calm water, represented by motion generated velocity potential. As the float size is small compared to wave length, diffraction potential is neglected. Added-mass and damping coefficients are computed from the motion generated velocity potential, using potential flow theory. In potential flow theory the flow can be characterised by a velocity potential function which satisfies the Laplace equation. Milgram (1976) and Garrison

(1978) have discussed the validity of this assumption in calculating wave loads on offshore structures.

The part of the wave energy which is supplied to the float by the incoming waves will be utilised to generate new waves. This motion generated disturbance is equal to the disturbance generated by the float with the same motion in otherwise calm water. The velocity potential of the disturbance at the surface ($z = 0$) may be written according to Lamb (1945) as

$$\phi_M(x,y,t) = \phi_M \cdot V_M = Re\{\phi_M(x,y)(-j\omega x_o e^{-j\omega t})\} \tag{1}$$

where

$$\begin{aligned} Re &= \text{"real part of"} \\ \phi_M &= \text{motion generated velocity potential} \\ V_M &= \dot{x} = -j\omega x_o e^{-j\omega t} \\ j &= \sqrt{-1} \\ \omega &= \text{radian frequency} \\ x_o &= \text{amplitude of float motion} \\ V_M &= \text{float velocity.} \end{aligned} \tag{2}$$

ϕ_M can be written as

$$\phi_M = \phi_{MC} + j\phi_{MS} \tag{3}$$

where ϕ_{MC} and ϕ_{MS} = real functions.

Using the linearised Bernoulli's equation and following Lee (1976), hydrodynamic force due to surge motion (F_M) can be expressed as

$$F_M = Re\left\{\rho\,\omega^2 x_o \int_{C_o} (\phi_{MC} + j\phi_{MS})\, n_1 \,\mathrm{d}s\, e^{-j\omega t}\right\} \tag{4}$$

where

$$\begin{aligned} \rho &= \text{sea water density} \\ c_o &= \text{cross-sectional contour of the float} \\ n_1 &= x\text{-component of unit normal vector} \\ \mathrm{d}s &= \text{elemental area.} \end{aligned}$$

Also, the motion-dependent force (F_M) is proportional to the float response and can be written in terms of added-mass and damping coefficients as

$$F_M = -(\alpha\ddot{x} + \beta\dot{x}) \tag{5}$$

where

$$\ddot{x} = -\omega^2 x_o e^{-j\omega t} = \text{float acceleration.}$$

Substituting $\dot{x}$ and $\ddot{x}$ in Equation (5)

$$F_M = Re\{(\omega^2\alpha + j\omega\beta)\,x_o e^{-j\omega t}\}\,. \tag{6}$$

Comparing Equations (5) and (6)

$$\alpha = \rho \int_{C_o} \phi_{MC}\, n_1\, \mathrm{d}s \tag{7}$$

$$\beta = \rho\omega \int_{C_o} \phi_{MS}\, n_1\, \mathrm{d}s\,. \tag{8}$$

Determination of added-mass and damping coefficients from the motion generated velocity potential is a boundary value problem and the solution is obtained by solving the boundary conditions using Green's function method. The boundary conditions are as follows:

(i) the classical Helmholtz equation,

$$\nabla^2 \phi_M + k^2 \phi_M = 0$$

where k = wave number $(2\pi/\lambda)$

(ii) for small amplitude motion, the linearized dynamic boundary condition at the free-surface,

$$\frac{\partial^2 \phi_M}{\partial t^2} + g\frac{\partial \phi_M}{\partial z} = 0 \qquad \text{on } z=0$$

(iii) the Sommerfeld radiation condition,

$$\lim_{r\to\infty} r^{\frac{1}{2}}\left(\frac{\partial \phi_M}{\partial r} - jk\phi_M\right) = 0$$

where $r = \sqrt{x^2 + y^2}$ is the radial distance from the origin, and

(iv) the boundary condition on the body surface,

$$\frac{\partial\, \phi_M}{\partial n} = n$$

where n is the unit normal vector.

3. ESTIMATION OF MOTION GENERATED VELOCITY POTENTIAL

From the geometry of Fig. 1, the motion generated velocity potential can be computed. Following the well-known Weber's solution (Harms, 1979), the motion generated velocity potential at any point on the float surface can be written as

$$\phi_M(Q_m(s)) = \frac{j}{4}\int_{C_o}\left(\phi_M(s)\frac{\partial H_o^1(kr)}{\partial n} - H_o^1(kr)\frac{\partial \phi_M(s)}{\partial n}\right)\mathrm{d}s + \phi_M(Q_m(s))\Big) \\ (\psi/2\pi) \tag{9}$$

where

$\phi_M(Q_m(s))$ ≡ motion generated velocity potential at a point Q_m on the surface

$\phi_M(s)$ ≡ motion generated velocity potential on the curve c_o.

$$H_o^1(kr) = J_o(kr) + jY_o(kr) \tag{10}$$

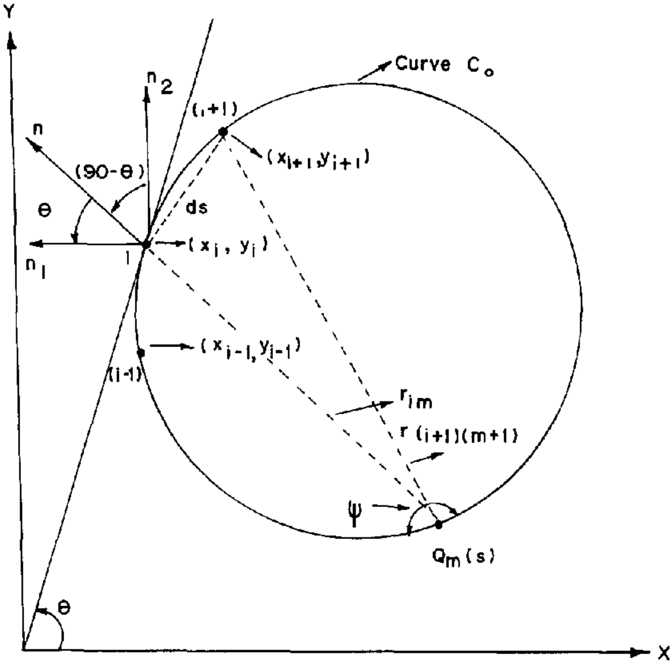

FIG. 1. Definition sketch for motion generated wave potential.

$\equiv$ Hankel function of the first kind and zeroth order

where

J_o $\equiv$ Bessel function of the first kind and zeroth order

Y_o $\equiv$ Bessel function of the second kind and zeroth order.

$$\frac{\partial H_o^1(kr)}{\partial n} = \sqrt{-J_1(kr) - jY_1(kr)} \tag{11}$$

where

J_1 $\equiv$ Bessel function of the first kind and first order

Y_1 $\equiv$ Bessel function of the second kind and first order

$r = \sqrt{(x_m - x_i)^2 + (y_m - y_i)^2} \equiv$ distance between Q_m and Q_i

ds = incremental arc length on the body curve c_o .

$$\frac{\partial \phi_M(s)}{\partial n} = n_1 = \sin\theta \tag{12}$$

and ψ = included angle at the point Q_m on the boundary.

Substituting Equations (3) and (10)–(12) in Equation (9) and separating into real and imaginary parts, we obtain

$$\phi_{MC}(Q_m(s))\left(1 - \frac{\psi}{2\pi}\right) = \frac{k}{4}\int_{C_o} (\phi_{MC}\, Y_1(kr) + \phi_{MS} J_1(kr)) r\, ds$$

$$+\frac{1}{4}\int_{C_o}(Y_o(kr)\sin\theta)\mathrm{d}s \qquad (13)$$

and

$$\phi_{MS}(Q_m(s))\left(1-\frac{\psi}{2\pi}\right)=\frac{k}{4}\int_{C_o}(\phi_{MS}\,Y_1(kr)-\phi_{MC}\,J_1(kr))r\,\mathrm{d}s$$

$$-\frac{1}{4}\int_{C_o}(J_o(kr)\sin\theta)\mathrm{d}s\,. \qquad (14)$$

The cross-section of the float is divided into equal segments by choosing points at equal intervals. The sum of the motion generated velocity potentials at these points gives the total potential, and this is resolved into real and imaginary parts, ϕ_{MC} and ϕ_{MS}, respectively. Using these estimated values of ϕ_{MC} and ϕ_{MS} in Equations (7) and (8), the added-mass and damping coefficients are computed.

4. RESULTS AND DISCUSSION

As the added-mass is in phase with the motion and damping coefficient out of phase with the motion, they greatly influence the motion characteristics of the tethered float. Variations of α and β with wave period (T) are shown in Fig. 2. The results of that figure show that α decreases rapidly with an increase in wave period up to 8 sec, and thereafter increases. But β initially increases and then decreases slowly with an increase in wave period. These variations indicate that both α and β are frequency dependent, and this is because motion generated velocity potential is associated with the generation of surface gravity waves by the oscillating float. The results in Fig. 3 show that both α and β increase gradually with respect to size of the float. Note: α and β do not change with water depth.

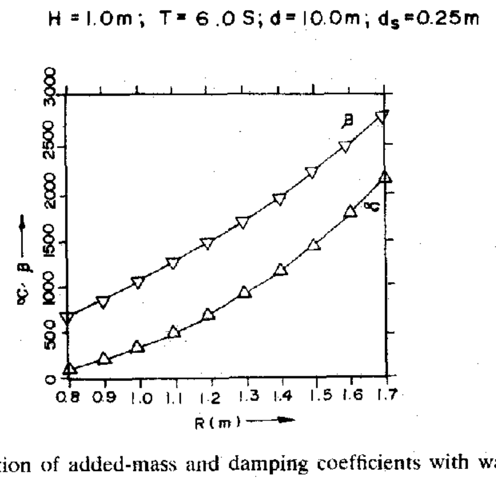

FIG. 2. Variation of added-mass and damping coefficients with wave period.

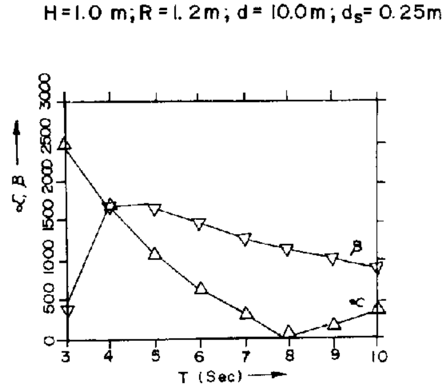

FIG. 3. Variation of added-mass and damping coefficients with float size.

The non-dimensionalised added-mass ($\bar{\alpha} = \alpha/m$) and damping coefficients ($\bar{\beta} = \beta/\omega m$) are both computed and presented in Fig. 4 to demonstrate their relative variations with respect to wave frequencies. Wave lengths (λ) in the range of 11–77 times the radius of the floats (λ = 14–92 m) are considered, as this is the range of wave lengths very often prevailing in coastal waters. Variations of $\bar{\alpha}$ and $\bar{\beta}$ are similar to those of α and β and it is significant to note that $\bar{\beta}$ remains constant for larger periods and dies out at lower periods.

A floating system experiences two types of damping, namely material and hydrodynamic. The material damping is generally small and neglected in the analysis. Therefore, the resonant response of the floating object is limited only by the amount of hydrodynamic damping present in the system. Furthermore, the natural and resonant frequencies are dependent on tether length (l) as well as added-mass and damping coefficients of

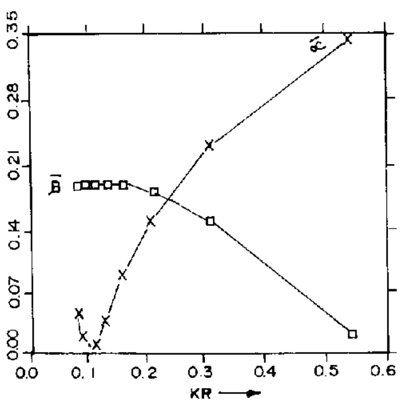

FIG. 4. Variation of non-dimensionalised added-mass and damping coefficients with wave length.

TABLE 1. NATURAL AND RESONANT PERIODS (d = 10.0 m, R = 1.2 m, H = 1.0 m, d_s = 0.25 m, l = 7.35 m)

T (sec)	$\bar{\alpha}$	$\bar{\beta}$	ω_n	T_n (sec)	ω_r	T_r (sec)
3	0.34	0.02	1.00	6.28	1.00	6.28
4	0.23	0.15	1.04	6.04	1.02	6.16
5	0.15	0.18	1.08	5.81	1.04	6.04
6	0.09	0.19	1.11	5.66	1.07	5.87
7	0.04	0.19	1.13	5.56	1.09	5.76
8	0.01	0.19	1.15	5.46	1.11	5.66
9	0.02	0.19	1.14	5.51	1.10	5.71
10	0.05	0.19	1.13	5.56	1.08	5.81

the float. The natural frequency (ω_n) of a tethered float whose mass is small with respect to the mass of displaced water is given by (Agerton *et al.*, 1976)

$$\omega_n = \sqrt{g/C_m \cdot l)} \tag{15}$$

where $C_m = 1 + \bar{\alpha}$

and resonant frequency (ω_r) by

$$\omega_r = \omega_n \sqrt{1 - 2\bar{\beta}^2}\,. \tag{16}$$

For a typical tethered float of radius 1.2 m and tether length 7.35 m, natural and resonant frequencies are computed for a wide range of wave periods. Natural and resonant periods (T_n and T_r) of oscillation of the float are presented in Table 1. The natural period of the float varies from 5.64 to 6.28 sec, and the resonant period from 5.66 to 6.28 sec. Both T_n and T_r are well within the limits of wave periods considered.

4.1. *Experimental results of natural period of floats*

Simple experiments were conducted in the wave flume to determine the natural period of oscillation of the float. The float was given a displacement in the horizontal direction and allowed to oscillate freely in a no-wave condition. From the number of oscillations and time duration, natural period was estimated. Similar tests were conducted for various float sizes, water depths and depths of submergence of floats. The results in Table 2 show that T_n increases with an increase in float size and tether length

TABLE 2. NATURAL PERIOD OF OSCILLATION OF FLOAT

Float size (cm)	Tether length (cm)	Water depth (cm)	Depth of submergence (cm)	Natural period (sec)
		117.5	2.5	1.73
15	90	122.5	7.5	1.70
		127.5	12.5	1.66
		132.0	2.0	2.00
20	100	136.5	6.5	1.83
		140.0	10.0	—
		131.0	1.0	2.10
30	90	137.0	7.0	2.00
		143.0	13.0	1.90

TABLE 3. COMPARISON OF WAVE FORCE CALCULATED FROM INCIDENT AND MOTION GENERATED VELOCITY POTENTIALS

$\bar{\beta}$	$\lvert f_w \rvert$	$\lvert f_w^1 \rvert$
0.15	15,924.98	17,846.90
0.18	18,200.44	19,550.30
0.19	19,559.62	20,086.02
0.19	20,414.39	20,090.80
0.19	20,981.34	20,091.10
0.19	21,374.71	20,091.10
0.19	21,658.09	20,091.10

and diminishes with increase in depth of submergence of floats. The natural period of the floats (prototype) varies from 4.4 to 6.4 sec for the wave periods 4.8–9.4 sec considered in the experimental study.

4.2. *Wave force and damping coefficient*

Newman (1962) has showed that a relationship exists between wave exciting force amplitude ($|f_w|$) and damping coefficient (β). For a two-dimensional problem this relationship can be written as

$$|f_w| = A_o \sqrt{\frac{\rho g^2}{\omega} \beta} \tag{17}$$

where A_o is the wave amplitude.

Applying Equation (17) to the present problem, one obtains

$$|f_w| = 3.5 A_o \sqrt{\frac{\rho g^2}{\omega} \beta} - |f_w^1| . \tag{18}$$

Using Equation (18), f_w^1 has been estimated and the values presented in Table 3. Wave-exciting forces calculated from incident wave potential (Vethamony *et al.*, 1991) as well as from damping coefficient show good agreement. This fact shows that wave excitation force can also be indirectly estimated from the motion generated velocity potential using the above relationship.

5. CONCLUSION

A method of estimating added-mass and damping coefficients from the motion generated velocity potential is presented. Variations of added-mass and damping coefficients for a single mode of motion (surge) have been studied with respect to wave period and float size. The effect of depth of submergence of floats on added-mass and damping coefficients has not been determined, which is a limitation of this study. A relationship is established between the wave exciting force amplitude and the damping coefficient of a tethered spherical float.

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REFERENCES

Agerton, D.J., Savage, G.H. and Stotz, K.C. 1976. Design, analysis and field test of a dynamic floating breakwater. *Proc. 15th Coastal Engng Conf.* **3**, 2792–2809.

Chung, J.S. 1976. Motion of floating structure in water of uniform depth. *J. Hydronautics* **10**, 65–73.

Chung, J.S. 1977. Forces on submerged cylinders oscillating near a free surface. *J. Hydronautics* **11**, 100–106.

Frank, W. 1967. Oscillation of cylinders in or below the free surface of deep fluids. Report No. 2375. NSRDC, Maryland, U.S.A.

Garrison, C.J. 1978. *Numerical Methods in Offshore Engineering*. John Wiley, New York (pp. 87–140).

Hanif, M. 1983. Analysis of heaving and swaying motion of a floating breakwater by finite element method. *Ocean Engng* **3**, 181–190.

Harms, V.W. 1979. Diffraction of water waves by isolated structures. *J. WatWay Port, Coastal Ocean Div.* **105**, 131–147.

Kim, W.D. 1965. On the harmonic oscillations of rigid body on a free surface. *J. Fluid Mech.* **21**, 427–451.

Lamb, H. 1945. *Hydronamics* (6th Edn). Dover, New York.

Lee, C.M. 1976. Motion characteristics of floating bodies. *J. Ship Res.* **20**, 181–189.

Milgram, J.H. 1976. Waves and wave forces. *BOSS '76, Trondheim*, Vol. 1, p. 11.

Newman, J.N. 1962. The exciting forces on fixed bodies in waves. *J. Ship Res.* **6**, 10–17.

Newton, R.E. 1975. *Finite Elements in Fluids*, Gallagher *et al.* (eds), Vol. 1, pp. 219–232.

Newton, R.E., Chenault, D.W. and Smith, D.A. 1974. Finite element solution for added mass and damping. *Proceedings of the International Symposium on Finite Element Methods in Flow Problems, Swansea, U.K.*

Vethamony, P. 1990. Motion and wave attenuation characteristics of a tethered float system. Ph.D. thesis submitted to the Indian Institute of Technology, Bombay (206 pp.).

Vethamony, P., Chandramohan, P. and Sastry, J.S. 1991. Estimation of wave force on a tethered spherical float using potential flow theory. Proceedings of the Third International Conference on Coastal and Port Engineering in Developing Countries, Mombasa, Kenya, September 1991.